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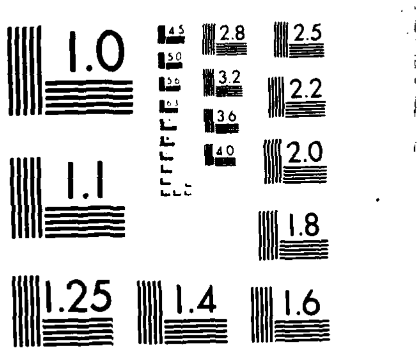

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FINDING THE LARGEST ℓ^p-BALL IN A POLYHEDRAL SET

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ABSTRACT

A simple linear programming formulation is given for finding an ℓ^p-ball with largest radius contained in a polyhedral set defined by m linear inequalities. The linear program also has m linear constraints similar to those defining the set. It is shown that finding the largest ball is not much more difficult than finding a feasible point. When the center of ball is fixed, the largest radius is easily obtained as the smallest of m ratios. The results can be extended to balls defined by other norms such as elliptic norms.

AMS (MOS) Subject Classifications: 90C05, 90C50

Key Words: linear programming, polyhedral set, ℓ^p-norm, computational complexity,

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SIGNIFICANCE AND EXPLANATION

Finding the largest ball or hypercube in a polyhedral set has many applications in operations research. This work gives a simple linear programming formulation for solving the problem. The effort needed to solve the linear program is almost the same as that for finding a feasible point in the given polyhedral set. Hence the result of this work can be considered optimal.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

FINDING THE LARGEST ℓ^p-BALL IN A POLYHEDRAL SET

Tzong-Huei Shiau

1. Introduction

We consider the problem of finding the largest ℓ^p-ball $B(y,\gamma;p)$ contained in a polyhedral set $F \subseteq R^n$, where

$$F = \{x \in R^n \mid a_i^T x \leq b_i \quad \text{for} \quad i = 1,2,\ldots,m\} \quad ,$$

$0 \neq a_i \in R^n$, $b_i \in R$, the superscript T denotes matrix transposition and $a_i^T x$ is therefore the inner product of a_i and x. For $y \in R^n$, $\gamma \in R$ and $1 \leq p \leq \infty$, the ℓ^p-ball with center y and radius γ is defined by

$$B(y,\gamma;p) := \{y + \gamma z \mid |z|_p \leq 1\} \quad \text{where}$$

$$(1) \qquad |z|_p := \begin{cases} \left(\sum_{i=1}^{n} |z_i|^p\right)^{1/p} & 1 \leq p < \infty \\ \max_i |z_i| & p = \infty \quad . \end{cases}$$

$B(y,\gamma;p)$ is an ordinary ball for $p = 2$. For $p = \infty$, it is a hypercube. Intuitively if F contains no interior points the largest ball will have $\gamma = 0$, otherwise $\gamma > 0$ (including the case $\gamma = +\infty$). Since the ℓ^p-ball is convex, $B(y,\gamma;p) \subseteq F$ if and only if F contains all the extreme points of $B(y,\gamma;p)$. In the case that $p = +\infty$, $B(y,\gamma;p)$ has 2^n extreme points,

$$y + \gamma z^k, \; k = 1,2,\ldots,2^n$$

where $z^k = [\pm 1,\pm 1,\ldots,\pm 1]^T$. Therefore the problem of finding the largest ball can be formulated as

$$(2) \qquad \max_{(y,\gamma)} \; \gamma \quad \text{subject to} \quad a_i^T(y + \gamma z^k) \leq b_i, \; i = 1,\ldots,m, \; k = 1,\ldots,2^n \quad .$$

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Although this is a linear program, it contains $2^n \cdot m$ constraints. Hence (2) is practically intractable even when n is as small as 20.

In this paper, we give a linear program formulation with only m constraints. The linear program (see (5)) is very similar to and no more difficult than the following formulation for finding a feasible point in F.

$$\max_{(y,\gamma)} \gamma \quad \text{subject to} \quad a_i^T y - \gamma \leq b_i \quad i = 1,\ldots,m \quad .$$

Hence the problem can be solved by efficient algorithms such as Dantzig's simplex method or, if the problem is large and sparse, Mangasarian's SOR method [6]. This result also shows that theoretically the problem is polynomial-time solvable [3], [4]. It is interesting to note that finding the smallest ball containing F is much more difficult. Depending on the norm used, it can be NP-complete [7], which means that if one can solve it in polynomial time then he can solve also in polynomial time hundreds of those <u>intractable</u> problems such as traveling salesman problem [2] or non-convex linear complementarity problems [1]. These problems are considered intractable because as it is widely believed, but not proven, that no polynomial-time algorithms exist for solving them.

2. LP Formulation

The problem can be written as

$$\text{(3)} \qquad \underset{(y,\gamma)}{\text{maximize}}\ \gamma \quad \text{subject to} \quad B(y,\gamma,p) \subseteq F$$

For $i = 1,2,\ldots,m$, define the function $g_i(y,\gamma;p)$ by

$$\text{(4)} \qquad g_i(y,\gamma;p) := \max_x a_i^T x - b_i \quad \text{subject to} \quad x \in B(y,\gamma;p) \quad .$$

It is easy to see that the constraint in (3) can be replaced by m constraints $g_i(y,\gamma;p) \leq 0 \quad i = 1,2,\ldots,m.$

<u>Lemma 1</u>: $B(y,\gamma;p) \subset F$ if and only if $g_i(y,\gamma;p) \leq 0$ for $i = 1,2,\ldots,m.$

<u>Proof</u>: $B(y,\gamma;p) \subset F$

iff

$$a_i^T x - b_i \leq 0 \quad \text{for all} \quad x \in B(y,\gamma;p) \quad \text{for} \quad i = 1,2,\ldots,m$$

iff

$$\left(\max_x \; a_i^T x - b_i \leq 0 \quad \text{subject to} \quad x \in B(y,\gamma;p)\right) \quad \text{for} \quad i = 1,2,\ldots,m$$

iff

$$g_i(y,\gamma;p) \leq 0 \quad \text{for} \quad i = 1,2,\ldots,m \quad .$$

□

<u>Lemma 2</u>. For $1 \leq p \leq \infty$, $g_i(y,\gamma;p) = a_i^T y - b_i + r \cdot |a_i|_q$ where $\frac{1}{p} + \frac{1}{q} = 1$.

<u>Proof</u>: For all $x \in B(y,\gamma;p)$, $|x-y| \leq \gamma$ and

$$a_i^T x - b_i = a_i^T y - b_i + a_i^T(x-y)$$

$$\leq a_i^T y - b_i + |a_i|_q \cdot |x-y|_p \leq a_i^T y - b_i + \gamma \cdot |a_i|_q \quad .$$

Hence $g_i(y,\gamma;p) \leq a_i^T y - b_i + \gamma \cdot |a_i|_q$. On the other hand, it is well-known that equalities hold for some $x \in B(y,\gamma;p)$. Hence $g_i(y,\gamma;p) = a_i^T y - b_i + r \cdot |a_i|_q$. To make the paper self-contained, we shall give the definition of x. Let a_{ij} denote the j-th component of a_i, define

$$\varepsilon_j = \begin{cases} +1 & \text{if} \quad a_{ij} \gtrless 0 \\ -1 & \text{if} \quad a_{ij} < 0 \end{cases} \qquad j = 1,2,\ldots,n \quad .$$

<u>Case $p = \infty$,</u> $q = 1$. $x := y + \gamma z$ where $z_j = \varepsilon_j$, $j = 1,\ldots,n$.

$$\text{So} \quad a^T(x-y) = \gamma \sum_{j=1}^{n} a_{ij} z_j = \gamma \sum_{j=1}^{n} |a_{ij}| = \gamma \cdot |a_i|_1 \; .$$

<u>Case $p = 1$,</u> $q = \infty$. $x := y + \gamma \varepsilon_k e_k$ where k is an index that $|a_i|_\infty = |a_{ik}|$ and e_k is the k-th unit vector.

<u>Case $1 < p < \infty$.</u> $x := y + \frac{\gamma}{|z|_p} z$ where $z_j = |a_{ij}|^{q-1} \varepsilon_j$.

So $|z|_p = \left(\Sigma \, |a_{ij}|^{(q-1)p}\right)^{1/p} = \left(\Sigma \, |a_{ij}|^q\right)^{1/p}$ since $(q-1) \cdot p = q$, and thereby,

$$a^T(x-y) = \frac{\gamma}{|z|_p} \cdot \Sigma|a_{ij}| \cdot |a_{ij}|^{q-1} = \gamma \cdot \frac{1}{|z|_p} \Sigma|a_{ij}|^q = \gamma \cdot (\Sigma|a_{ij}|^q)^{1-1/p}$$

$$= \gamma \cdot (\Sigma|a_{ij}|^q)^{1/q} = \gamma \cdot |a_{ij}|_q \quad .$$

□

It follows that (3) is equivalent to

(5) $$\underset{(y,\gamma)}{\text{maximize}}\ \gamma \quad \text{subject to} \quad a_i^T y + \gamma \cdot |a_i|_q \leq b_i, \ i = 1,2,\ldots,m \quad .$$

We summarize the results in the following:

<u>Theorem 1</u>. For any given a_i, b_i, $i = 1,2,\ldots,m$

(i) The linear program (5) is feasible.

(ii) Assume that the linear program (5) is bounded, and that (y^*,γ^*) is an optimal solution with $\gamma^* < +\infty$. Then

(a) $\gamma^* < 0$ if and only if $F = \phi$.

(b) $\gamma^* = 0$ if and only if $F \neq \phi$ but F has empty interior.

(c) $\gamma^* > 0$ if and only if $F^o \neq \phi$ and $B(y^*,\gamma^*;p)$ is an ℓ^p-ball contained in F with the largest radius.

<u>Proof</u>. Since (5) is equivalent to (3) by Lemma 1, 2. The theorem follows by the following observations.

(i) $(0,-K)$ is feasible for K sufficiently large.

(ii) $(y,0)$ is feasible for all $y \in F$.

(iii) F has non-empty interior if and only if F contains a ball $B(y,\gamma;p)$ with $\gamma > 0$.

□

<u>Remarks</u> (i) If $\gamma^* = +\infty$ then F has unbounded interior. But the reverse is not true. For example, let

$$F = \{(x_1,x_2)|x_1 - x_2 \leq 1, -x_1 + x_2 \leq 0\} \quad .$$

The same example also shows that y^* is not unique.

(ii) Given $y \in F$, the largest ball contained in F and centered at y can be found by solving

$$\max_{\gamma} \gamma \quad \text{s.t.} \quad \gamma \cdot |a_{ij}|_q \leq b_i - a_i^T y, \quad i = 1,\ldots,m$$

which can be solved explicitly, namely,

$$\gamma^* = \min_{1 \leq i \leq m} \frac{b_i - a_i^T y}{|a_i|_q} \quad . \tag{6}$$

3. General Norms

From the derivation of (5) in the previous section, it is clear that the results can be generalized to other norms. That is, find a largest ball $B(y,\gamma) = \{x \mid |x-y| \leq \gamma\}$ in F is equivalent to

$$\max_{(y,\gamma)} \gamma \quad \text{s.t.} \quad a_i^T y + \gamma \cdot |a_i|^* \leq b_i, \quad i = 1,\ldots,m \tag{7}$$

where $|a_i|^*$ is the dual norm of $|\cdot|$, namely

$$|a_i|^* = \max a_i^T z \quad \text{s.t.} \quad |z| \leq 1 \quad . \tag{8}$$

Of course, to make (7) useful computationally, we need to be able to solve (8), as in Section 2. For example, if $|z| := (z^T A z)^{1/2}$ *where* A *is a symmetric positive definite* matrix, then by Kuhn-Tucker Theorem [5] we have

$$\begin{aligned} |a_i|^* &= \max_{z} a_i^T z \quad \text{s.t.} \quad z^T A z \leq 1 \\ &= (a_i^T A^{-1} a_i)^{1/2} \quad . \end{aligned} \tag{9}$$

Note that in this case, a "ball" is in fact an ellipsoid. Hence one can find the largest ellipsoid (with a given shape defined by A) in F by solving a linear program. When the center is fixed, the largest ellipsoid can easily be found by (6) in which $|a_i|_q$ is replaced by $(a_i^T A^{-1} a_i)^{1/2}$.

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